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Asymptotic expansions for laminar forced-convection heat and mass transfer

Part 1. Low speed flows

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A method is presented in this article for deriving higher-order correction terms to the well-known asymptotic results for laminar forced-convection heat and mass transfer, and a formula is obtained for computing under fairly general conditions the first correction term to the asymptotic Nusselt number at large Péclet numbers for flows with small or moderate Reynolds numbers. This result is then applied to the problem of heat transfer from a solid, isothermal sphere in Stokes flow, to yield the asymptotic expression for the average Nusselt number,

$$\overline{Nu} = (Pe)^{\frac{1}{3}}[0{\cdot}6245 + 0{\cdot}461(Pe)^{-\frac{1}{3}} + O(Re) + o(Pe^{-\frac{1}{3}})]$$

for $Pe \to \infty$, $Re \to 0$, where $\overline{Nu}$ and Pe are based on the radius of the sphere.

1. Introduction

Some of the most useful mathematical results for laminar forced and natural convection heat or mass transfer are based on asymptotic solutions of the governing transport equations. Familiar examples include the laminar forced-convection heat-transfer relations for the Nusselt number in terms of the Prandtl and Reynolds numbers, such as

$$Nu \propto (Pr\,Re)^{\frac{1}{3}}$$

for low Reynolds number laminar flows (Acrivos & Taylor 1962), and

$$Nu \propto (Pr)^{\frac{1}{3}}(Re)^{\frac{1}{2}} \quad \text{or} \quad Nu \propto (Pr\,Re)^{\frac{1}{2}},$$

depending on whether $Pr \gg 1$ or $Pr \ll 1$, for high-speed flows of the laminar boundary-layer type (Morgan & Warner 1956; Morgan, Pipkin & Warner 1958).

It has already been pointed out in several of the papers on this subject (Lighthill 1950; Meksyn 1961; Mercer 1959, 1960; Merk 1959; Morgan & Warner 1956; Morgan *et al.* 1958) that such relations represent, as a rule, asymptotic results which become exact only for very large or for very small values of certain of the parameters involved. Nevertheless, these asymptotic solutions are exceedingly useful from a practical point of view owing to the fact that in many

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cases they have been found to hold with surprising accuracy even under distinctly non-asymptotic conditions. It would appear desirable therefore to develop a general technique for constructing corrections to these asymptotic solutions in order to be able to estimate theoretically their range of validity for general flow fields, surface geometries and boundary conditions.

Now, it is plausible to suppose that when relations such as those given above are truly of an asymptotic type representing, say, the first term in an asymptotic series, it sould be possible to generate 'correction terms' by means of a straightforward perturbation scheme. Indeed, this has already been carried out for certain special cases (Meksyn 1961; Mercer 1959, 1960; Merk 1959; Morgan & Warner 1956; Morgan *et al.* 1958). In this paper we shall extend this approach to laminar forced convection at high Péclet numbers, but with small or moderate Reynolds numbers, and shall derive a general formula for computing the first-order correction term to the asymptotic form of the Nusselt number for $Pe = (Re\,Pr) \to \infty$ which will then be applied to the problem of heat transfer from an isothermal sphere in Stokes flow. Some expansions for heat transfer to laminar boundary-layer flows with large or with small Prandtl numbers will be treated in a subsequent paper.

2. Basic equations and the boundary-value problem

We begin with the well-known steady-state energy equation for constant property incompressible flows. In dimensionless form this becomes

$$M(\Theta) \overset{\text{def}}{=} \mathbf{U}.\operatorname{grad}\Theta - (1/Pe)\,\nabla^2\Theta = H, \tag{2.1}$$

where Θ denotes a dimensionless temperature relative to that of the free stream, while

$$Pe = U_\infty L/\kappa \tag{2.2}$$

is a (dimensionless) Péclet number, U_∞ being a characteristic velocity, L a characteristic length for the system and κ the thermal diffusivity of the fluid. Also, $\mathbf{U}$ is the velocity vector and H a heat-generation term, both of which are assumed known everywhere throughout the flow field. The analysis will be restricted to two-dimensional velocity and temperature fields, but will include axisymmetric as well as planar flows past a solid bounding surface $\mathscr{S}$.

We shall assume further that the surface $\mathscr{S}$ is sufficiently smooth, except perhaps at a finite number of isolated points, so that we may employ the usual 'boundary-layer' co-ordinates as independent variables in a region near the surface $\mathscr{S}$. Thus, letting

$$x = x_1/L, \quad y = x_2/L,$$

where x_2 is the perpendicular distance of a point from the surface and x_1 is a distance measured along the surface in a plane of flow, we can write equation (2.1), at least near the surface $\mathscr{S}$, as

$$P(\Theta) = Q, \tag{2.3}$$

where P denotes the linear differential operator

$$P = \alpha\gamma M = \gamma u\frac{\partial}{\partial x} + \alpha\gamma v\frac{\partial}{\partial y} - \frac{1}{Pe}\left[\frac{\partial}{\partial x}\left(\frac{\gamma}{\alpha}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha\gamma\frac{\partial}{\partial y}\right)\right].$$

In this expression, $u = u_1/U_\infty$ and $v = u_2/U_\infty$, u_1 and u_2 being the velocity components in the x_1, x_2 direction, respectively, and

$$\left.\begin{aligned} \alpha &= 1+\alpha_1(x)\,y, \\ \gamma &= \gamma_0(x)+\gamma_1(x)\,y, \\ Q(x,y) &= \alpha\gamma H(x,y), \end{aligned}\right\} \tag{2.4}$$

where α and γ are the metrical coefficients for the co-ordinate system (Meksyn 1961). We recall that $\alpha_1(x)/L$ is the curvature of the surface in the plane of flow for both the planar and the axisymmetric cases. Also, for planar flow $\gamma_0 = 1$, $\gamma_1 = 0$, whereas, for axisymmetric flow, $L\gamma_0$ and γ_1 are, respectively, the radius of rotation of the surface $\mathscr{S}$ and the cosine of the angle between the tangent to the surface and the axis of symmetry. Thus, for axisymmetric flow, we may set

$$\gamma_1 = \pm[1-(\gamma_0')^2]^{\frac{1}{2}}$$

and

$$\alpha_1 = \gamma_1'/\gamma_0' = -\gamma_0''/\gamma_1, \tag{2.5}$$

where the primes denote differentiation with respect to x and where the sign in the expression for γ_1 is to be taken positive or negative according to whether $\cos^{-1}\gamma_1$ is $\lessgtr \frac{1}{2}\pi$.

We shall consider here the boundary-value problem in which the temperature is prescribed on the surface $\mathscr{S}$, i.e.

$$\lim_{y\to 0} \Theta = \Theta_s(x), \tag{2.6}$$

where $\Theta_s(x)$ is given *a priori*. For problems of the exterior type, where the surface $\mathscr{S}$ bounds the flow field internally, an additional condition of regularity far from the surface will be necessary, and we shall suppose here that this can be stated as

$$\lim_{y\to\infty} \Theta = 0, \tag{2.7}$$

it being assumed that H behaves in such a way that a solution exists which is regular at $y = \infty$.

As is generally recognized, it is not feasible to derive exact solutions to the above convection problem for the most general flow field and for any given bounding surface $\mathscr{S}$. One is forced therefore to resort to approximate or, at best, to asymptotic solutions such as the thermal boundary-layer solution to be considered here, which may be regarded as the limiting form of the temperature field Θ for large values of the Péclet number.

One recalls at this point that, for $Pe \to \infty$, temperature variations are in general confined to a small region near the surface $\mathscr{S}$, in which case the rate of convection is determined primarily by the limiting form near $y = 0$ of the functional coefficients α, γ, u, v appearing in the differential operator P of (2.3). Now, for a given surface geometry, the functions α, γ are completely specified, whereas the character of the velocity components u, v is governed by the surface geometry and by a Reynolds number

$$Re = U_\infty L/\nu, \tag{2.8}$$

ν being the kinematic viscosity of the fluid. Therefore, since

$$Pe = Re\,Pr,$$

where

$$Pr = \nu/\kappa \tag{2.9}$$

is the Prandtl number of the fluid, we can state formally that the limit $Pe \to \infty$ includes the special cases:

$$\left.\begin{array}{l}\text{(1) } Re \text{ fixed with } Pr \to \infty;\\ \text{(2) } Re \to \infty \text{ with } Pr \text{ fixed.}\end{array}\right\} \tag{2.10}$$

The first of these corresponds to fairly general laminar flows including those with very small Reynolds numbers, whereas the requirement of large Re in the second case implies that the flow is of the laminar boundary-layer type.

It will be shown in the present paper how, for case (1), a higher-order correction to the thermal boundary-layer solution can be derived, while any discussion of the interesting subcases of case (2), $Pr \to 0$ and $Pr \to \infty$, which can also be treated by the same type of analysis as that to be presented below, will be deferred to a later article.

3. The thermal boundary-layer expansion for small or moderate Reynolds numbers

Here we shall outline briefly the type of boundary-layer expansion to be considered, establishing at the same time the notation to be employed.

First of all, since we are dealing with incompressible flow, we can set

$$u = \frac{1}{\gamma}\frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{\alpha\gamma}\frac{\partial \psi}{\partial x}, \quad (\psi = 0 \quad \text{at} \quad y = 0), \tag{3.1}$$

where ψ is the stream function, in which case the differential operator P of (2.3) becomes

$$P = \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{1}{Pe}\left[\frac{\partial}{\partial x}\left(\frac{\gamma}{\alpha}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha\gamma\frac{\partial}{\partial y}\right)\right]. \tag{3.2}$$

Now, as has already been remarked, the asymptotic expression for P as $Pe \to \infty$ depends, primarily, on the limiting form of its coefficients as $y \to 0$. Thus, if the analytic behaviour of the functions α and γ is restricted by adopting the usual assumptions of boundary-layer analysis that the surface $\mathscr{S}$ is 'smooth', i.e. that it has a finite curvature ($|\alpha_1(x)| < \infty$), and that in the axisymmetric case its radius of revolution is not zero ($\gamma_0(x) > 0$), we may expand both the coefficients $\alpha\gamma$ and γ/α, occurring in P, about $y = 0$, so that

$$\left.\begin{aligned}\alpha\gamma &= \gamma_0 + (\alpha_1\gamma_0 + \gamma_1)\,y + \gamma_1\alpha_1 y^2,\\ \gamma/\alpha &= \gamma_0 + (\gamma_1 - \gamma_0\alpha_1)\,y + (\alpha_1^2\gamma_0 - \alpha_1\gamma_1)\,y^2 + \ldots,\end{aligned}\right\} \tag{3.3}$$

in which the zeroth-order coefficient does not vanish and where the higher-order coefficients are bounded; moreover, for a solid surface, the stream function will have in general the expansion

$$\psi = \psi_2(x)\,y^2 + \psi_3(x)\,y^3 + \ldots, \tag{3.4}$$

where

$$\psi_2(x) = \tfrac{1}{2}\gamma_0\frac{\partial u}{\partial y}(x, 0). \tag{3.5}$$

Thus, as long as $\psi_2(x)$, which is proportional to the shear stress at the surface, does not vanish, the stream function ψ has a zero of order *two* in y at $y = 0$. The appropriate stretched co-ordinate for the boundary-layer analysis is then, in this case (Morgan & Warner 1956; Levich 1962)

$$y_2 = (Pe)^{\frac{1}{3}} y,$$

in terms of which (3.3) and (3.4) become

$$\left.\begin{aligned} \alpha\gamma &= \gamma_0 + (\alpha_1\gamma_0 + \gamma_1)(Pe)^{-\frac{1}{3}} y_2 + O(Pe^{-\frac{2}{3}}), \\ \gamma/\alpha &= \gamma_0 + (\gamma_1 - \gamma_0\alpha_1)(Pe)^{-\frac{1}{3}} y_2 + O(Pe^{-\frac{2}{3}}), \\ \psi &= Pe^{-\frac{2}{3}}[\psi_2 y_2^2 + \psi_3 (Pe)^{-\frac{1}{3}} y_2^3 + O(Pe^{-\frac{2}{3}})]. \end{aligned}\right\} \tag{3.6}$$

As a result, the operator P of (3.2) can be expanded formally as

$$P = (Pe)^{-\frac{1}{3}}[P_0 + (Pe)^{-\frac{1}{3}} P_1 + O(Pe^{-\frac{2}{3}})], \tag{3.7}$$

where
$$P_0 = 2\psi_2 y_2 \frac{\partial}{\partial x} - \psi_2' y_2^2 \frac{\partial}{\partial y_2} - \gamma_0 \frac{\partial^2}{\partial y_2^2},$$

$$P_1 = 3\psi_3 y_2^2 \frac{\partial}{\partial x} - \psi_3' y_2^3 \frac{\partial}{\partial y_2} - (\gamma_1 + \gamma_0\alpha_1)\frac{\partial}{\partial y_2}\left(y_2 \frac{\partial}{\partial y_2}\right),$$

etc., are linear differential operators containing no explicit dependence on Pe.

We shall consider henceforth only the homogeneous form of (2.3), with $Q = 0$, since the analysis which follows could easily be extended to the non-homogeneous equation if the analytic form of Q were specified. (In this way, for example, one could account for viscous energy dissipation in the flow.) The solution to the homogeneous equation can then be constructed formally by letting

$$\Theta = \Theta_0(x, y_2) + (Pe)^{-\frac{1}{3}}\Theta_1(x, y_2) + O(Pe^{-\frac{2}{3}}), \tag{3.8}$$

where the Θ_i are functions only of x and y_2 for $i = 0, 1, 2, \ldots$. In particular, in view of (3.7) and (3.8),

$$P\Theta = (Pe)^{-\frac{1}{3}}[P_0\Theta_0 + (P_0\Theta_1 + P_1\Theta_0)(Pe)^{-\frac{1}{3}} + O(Pe^{-\frac{2}{3}})] = 0,$$

which, on equating to zero the coefficients of like powers of Pe, reduces to the sequence of differential equations

$$\left.\begin{aligned} P_0\Theta_0 &= 0, \\ P_0\Theta_1 &= -P_1\Theta_0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{3.9}$$

Similarly, the boundary conditions of (2.6) and (2.7) become

$$\left.\begin{aligned} &\lim_{y_2\to 0} \Theta_i = \begin{cases} \Theta_s(x), & \text{for} \quad i = 0 \\ 0, & \text{for} \quad i = 1, 2, \ldots, \end{cases} \\ &\lim_{y_2\to\infty} \Theta_i = 0, \qquad \text{for} \quad i = 0, 1, 2. \end{aligned}\right\} \tag{3.10}$$

The first equation of (3.9) represents, of course, the 'boundary-layer' approximation to the energy equation which has found frequent applications. We now wish to improve on this approximation by deriving an expression for the first correction with the aid of the appropriate Green's function for the differential operator P_0. This can be done in principle since the preceding system of equations constitutes a typical perturbation scheme which is formally soluble in a sequential fashion if an appropriate initial condition is imposed on the Θ_i. Such a condition is of course necessary for uniqueness since we have replaced the *elliptic* partial differential equation (2.3) by a sequence of *parabolic* equations.

This initial condition must, however, be specified with some care. To begin with, it appears reasonable that, for $i = 0, 1, 2, \ldots, \Theta_i \to 0$ for $x \to 0+$ along any streamline $\psi > 0$, in which case, if we consider only the first term of (3.6) for ψ, we should require that

$$\lim_{x\to 0} \Theta_i = 0, \quad \text{for all} \quad \psi_2(x)\, y_2^2 > 0, \quad (i=0,1,2,\ldots), \tag{3.11}$$

the point $x = 0$ corresponding to the leading edge or forward stagnation point of the surface. This is equivalent to neglecting any initial thermal boundary-layer profile, an assumption which is permissible certainly to $O(Pe^{-\frac{1}{3}})$ as long as $\Theta_s(x)$ is both analytic and such that $\lim_{x\to 0} \Theta_s(x) = 0$. In contrast, if $\lim_{x\to 0} \Theta_s(x) \neq 0$ then the conduction of heat upstream from the leading edge cannot be overlooked and any correct analysis of the problem must take into account, in general, the presence of this singularity at $x = 0$. In particular, if one supposes that as $x \to 0$, $\psi_2(x) \sim \alpha x^m$ with $0 \leqslant m < 1$, then one can show that (3.8) can only hold as long as $x > O(1/Pe^{1/(m+2)})$, while within the singular region $|x| < O(1/Pe^{1/(m+2)})$,

$$2\alpha X^m Y \frac{\partial \Theta}{\partial X} = \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2}, \tag{3.11a}$$

with $Y = Pe^{1/(m+2)}y$, $X = Pe^{1/(m+2)}x$, together with the surface boundary conditions,

$$\text{at } Y = 0, \quad \Theta = \Theta_s(0) \quad \text{for} \quad X \geqslant 0 \quad \text{and} \quad \partial\Theta/\partial Y = 0 \quad \text{for} \quad X < 0.$$

It can be easily demonstrated, furthermore, that under these conditions the contribution of the singular region to the surface integral of the local heat flux $(\partial\Theta/\partial y)_{y=0}$ becomes $O(1)$, which is clearly of the same order of magnitude as that of the second term of (3.8).

Thus, it should be firmly kept in mind that the formal expansion to be developed below cannot be applied to problems in which the heating section is allowed to extend all the way up to the leading edge. The only exception to this restriction appears to be the case of symmetric stagnation flows with symmetric surface temperature distributions where, as will be remarked on later, the exact initial condition $(\partial\Theta/\partial x) = 0$ as $x \to 0$ is satisfied by at least the first two terms of (3.8) even if $\Theta_s(0) \neq 0$.

Actually, in the discussion which follows, we shall consider a slightly more general expansion problem than that outlined above by allowing the stream function ψ to have a zero of arbitrary positive order $n > 0$ at $y = 0$, such that

$$\psi \sim \psi_n(x)\, y^n \quad \text{for} \quad y \to 0, \tag{3.12}$$

in which case the appropriate stretched co-ordinate for the boundary-layer analysis becomes (Acrivos 1960)

$$y_n = (Pe)^{1/(n+1)}\, y, \tag{3.13}$$

while the resulting asymptotic series for Θ assumes the form

$$\Theta = \Theta(x, y_n) + (Pe)^{-1/(n+1)}\,\Theta_1(x, y_n) + O(Pe^{-2/(n+1)}) \tag{3.14}$$

involving now a formal set of equations

$$\left.\begin{aligned}
&P_0\,\Theta_i = \begin{cases} 0 & \text{for} \quad i = 0 \\ Q_i(x, y_n) & \text{for} \quad i = 1, 2, \ldots, \end{cases} \\
&\lim_{y_n \to 0} \Theta_i = \begin{cases} \Theta_s(x) & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i = 1, 2, \ldots, \end{cases} \\
&\lim_{y_n \to \infty} \Theta_i = 0 \qquad\qquad \text{for} \quad i = 0, 1, 2, \ldots, \\
&\lim_{x \to 0} \Theta_i \ \ = 0 \qquad\qquad \text{for} \quad i = 0, 1, 2, \ldots, \psi_n(x)\, y_n^n > 0.
\end{aligned}\right\} \tag{3.15}$$

Here, the Q_i are determined once Θ_0, Θ_1, Θ_{i-1} are given for $i = 1, 2, \ldots$, and P_0 is the differential operator

$$P_0 = n\psi(x)\, y^{n-1}\frac{\partial}{\partial x} - \psi'(x)\, y^n \frac{\partial}{\partial y} - \gamma_0 \frac{\partial^2}{\partial y^2}, \tag{3.16}$$

where, for convenience, we have dropped the subscript n on y_n and $\psi_n(x)$. This generalization to arbitrary n could have practical applications to heat-transfer problems involving transfer at the interface between two fluids ($\psi_1 \neq 0$, $n = 1$) or from surfaces where the shear stress vanishes ($n = 3$).

In closing this section it is important to point out that one should not expect the *regular* perturbation scheme outlined above to remain valid under all circumstances since one can find examples, such as those to be mentioned below, where this procedure clearly cannot be applied. Thus, it is evident that the method given by (3.7) and (3.8) would fail if $\psi_2(x)/\psi_3(x)$ were to vanish at one or more isolated points along the surface while remaining non-zero everywhere else. Similarly, the scheme would cease to apply in regions where the thickness of the thermal layer remains of the same order of magnitude as the characteristic dimension of the body. And finally, this approach cannot remain valid if the surface temperature distribution $\Theta_s(x)$ has a finite discontinuity at at least one point $x = \xi$, since one can easily show that the temperature gradients will be of the same order of magnitude near ξ both in the longitudinal and in the normal directions. Consequently, in order to take proper account of the discontinuity at $x = \xi$ one would have to include in the analysis a treatment of an equation similar to (3.11*a*), usually with $m = 0$, for the region near $x = \xi$.

4. Construction of a formal solution to the expansion problem

4.1. *Transformation of co-ordinates and introduction of the Green's function*

By generalizing the transformation used by Herbeck (1954) for planar flows in a related problem, we now change to new independent variables

$$\left.\begin{aligned}
x &= [n\psi(x)]^{1/n}\, y, \\
z &= \int_0^x [n\psi(s)]^{1/n}\, \gamma_0(s)\, ds,
\end{aligned}\right\} \tag{4.1}$$

where we suppose that $n > 0$, $\gamma_0 > 0$, and $\psi > 0$ for $x > 0$, and that the defining integral for t exists. For external flows, the point $x = 0$ ($t = 0$) refers to the leading edge or forward stagnation point of the surface under consideration.

In terms of these new variables, the operator P_0 of (3.16) becomes

$$P_0 = J^{-1}[z^{n-1}\partial/\partial t - (\partial^2/\partial z^2)], \tag{4.2}$$

where J denotes the Jacobian of the transformation

$$J = \partial(x,y)/\partial(t,z) = 1/\gamma_0(x)\,[n\psi(x)]^{2/n}. \tag{4.3}$$

This operator in many of its equivalent forms has already been the subject of a number of mathematical and physical papers. The more mathematical of these, Myller-Lebedeff (1907), Hille (1926) and Hardy (1932), proceed from various forms of P_0 and derive fundamental solutions to the corresponding homogeneous differential equation, some of which have already been applied to physical problems by Sutton (1943) and by Lauwerier (1959) for the case $n = 2$.† For $n = 1$ the operator P_0 reduces to that associated with unsteady unidimensional heat conduction or diffusion.

Although some of the fundamental solutions given by Sutton correspond to solutions previously reported, the various theorems and results cited in his paper are most directly applicable to the problem treated here (after a simple co-ordinate transformation). Consequently, we shall employ these results in a formal way referring the interested reader to his paper for some of the proofs involved.

We begin by expressing formally the solution to the boundary-value problem

$$P_0\Theta = q(t,z) \quad \text{for} \quad t > 0, z > 0, \tag{4.4}$$

$$\lim_{t\to 0} \Theta = f(z) \quad \text{for} \quad z > 0, \tag{4.5}$$

$$\lim_{z\to 0} \Theta = h(t) \quad \text{for} \quad t > 0, \tag{4.6}$$

by

$$\Theta = \Xi(t,z) + \chi(t,z) + \Lambda(t,z), \tag{4.7}$$

where

$$\Xi = \int_0^\infty z^{*n-1} G_0(t,z;\,0,z^*)\,f(z^*)\,dz^*,$$

$$\chi = \int_0^t \left[\frac{\partial G_0}{\partial z^*}(t,z;\,t^*,z^*)\right]_{z^*=0} h(t^*)\,dt^*,$$

$$\Lambda = \int_0^t \int_0^\infty G_0(t,z;\,t^*,z^*)\,J(t^*)\,q(t^*,z^*)\,dz^*\,dt^*,$$

J denoting the Jacobian of (4.2) and $G_0(t,z;t^*,z^*)$ the Green's function, the latter satisfying (Friedman 1956)

$$\left.\begin{aligned} P_0 G_0 &= J^{-1}\,\delta(t-t^*)\,\delta(z-z^*),\\ \lim_{t\to t^*} G_0 &= 0,\\ \lim_{z\to 0} G_0 &= 0, \end{aligned}\right\} \tag{4.8}$$

where δ is the Dirac delta function.

† Lighthill (1950), in his treatment of heat transfer to laminar boundary layers with large Prandtl number, and LeFur (1960) have made use of the Laplace transformation to treat boundary-value problems associated with the above operator.

Now, the appropriate form for G_0 is found from Sutton's results to be

$$G_0 = \begin{cases} \dfrac{(zz^*)^{\frac{1}{2}}}{n+1} \exp\{-(\zeta^2+\zeta^{*2})\} I_{1/(n+1)}(2\zeta\zeta^*), & \text{for} \quad \tau > 0, \\ 0, \quad \text{for} \quad \tau \leqslant 0, \end{cases} \tag{4.9}$$

where
$$\tau = t - t^*,$$
$$\zeta = \zeta(z,\tau) = z^{\frac{1}{2}+\frac{1}{2}n}/(n+1)\,\tau^{\frac{1}{2}},$$
$$\zeta^* = \zeta(z^*,\tau).$$

$I_{1/n+1}$ denotes the modified Bessel function of the first kind of order $1/n+1$.

The functions Ξ and χ in (4.7) satisfy the homogeneous form of (4.4) and Λ the non-homogeneous form, with χ and Λ satisfying the homogeneous form of the initial condition (4.5) and Ξ the non-homogeneous form; finally, Ξ and Λ satisfy the homogeneous form of the boundary condition (4.6) and χ the non-homogeneous form.

4.2. *Application to the general expansion problem*

The boundary-value problem of (3.15) is seen to be a special case of (4.4)–(4.6). In particular, since we shall restrict ourselves to problems for which $f(z) \equiv 0$ in (4.5), and therefore $\Xi \equiv 0$, we need only consider integrals of the type χ and Λ in (4.7).

The first perturbation function Θ_0 of (4.4), i.e. the boundary-layer solution, is given immediately by the second integral of (4.7):

$$\left.\begin{aligned} \Theta_0 &= \chi(t,z), \\ \text{with} \quad h(t) &= \Theta_s(x). \end{aligned}\right\} \tag{4.10}$$

However, since we are dealing with linear equations, it suffices to consider the simpler problem in which

$$h(t) = \Theta_s(x) = H(x-x^*) = H(t-t^*), \tag{4.11}$$

where $H(s)$ is the Heaviside step-function, being zero or unity according to whether $s < 0$ or $s > 0$. The solution to the problem involving a more general variation of Θ_s can then be derived in general by superposition from results thus obtained.

Assuming then that $h(t)$ is indeed given by (4.11), we can reduce the integral χ for Θ_0 to

$$\Theta_0 = \frac{\Gamma[1/(n+1), \zeta^2]}{\Gamma[1/(n+1)]}, \tag{4.12}$$

where ζ is the variable defined in (4.9), and $\Gamma(\nu,s)$ is the complement of the tabulated, incomplete gamma function:

$$\Gamma(\nu,s) = \int_s^\infty e^{-\lambda}\lambda^{\nu-1}\,d\lambda.$$

This solution appears in Sutton's paper and has also been obtained by Acrivos (1960) by a 'similarity' transformation.

A formal solution for the second perturbation, or first correction term, Θ_1, in (4.14) can now be expressed in terms of the third integral of (4.7) as

$$\left.\begin{aligned}\Theta_1 &= \Lambda(t,z),\\ \text{with}\qquad q(t,z) &= Q_1(x,y).\end{aligned}\right\} \tag{4.13}$$

Before going any further, however, it is necessary to specify the functional form of $Q_1(x,y)$ in (3.15). Of particular interest here is the perturbation scheme of (3.8)–(3.10) with $n = 2$, for which one finds, using (4.1), that

$$\begin{aligned}q(t,z) &= -P_1\Theta_0\\ &= -\frac{3\psi_3}{(2\psi_2)^{\frac{1}{2}}}\gamma_0 z^2\frac{\partial\Theta_0}{\partial t}+\left[\frac{3\psi_3\psi_2'}{2\psi_2}-\psi_3'\right]\frac{z^3}{2\psi_2}\frac{\partial\Theta_0}{\partial z}-(\gamma_1+\alpha_1\gamma_0)(2\psi_2)^{\frac{1}{2}}\frac{\partial}{\partial z}\left(z\frac{\partial\Theta_0}{\partial z}\right),\end{aligned}$$

from which it follows that

$$J(t)\,q(t,z) = -\frac{3^{\frac{2}{3}}\,e^{-\zeta^2}}{\Gamma(\frac{1}{3})\,(3\tau)^{\frac{1}{3}}}[A^{(0)}(\tau,t^*)+A^{(3)}(\tau,t^*)\,(3\zeta)^2], \tag{4.14}$$

by substituting the function Θ_0 as given by (4.12). Here $\zeta = z^{\frac{3}{2}}/3\tau^{\frac{1}{2}}$ and τ are the variables defined in (4.9) for $n = 2$, whereas $J(t)$ is the Jacobian of (4.2), and

$$\left.\begin{aligned}A^{(0)}(\tau,t^*) &= \{1/(2\psi_2)^{\frac{1}{2}}\}(\alpha_1+\gamma_1/\gamma_0),\\ A^{(3)}(\tau,t^*) &= \frac{3\psi_3}{\gamma_0(2\psi_2)^2}\tau\left[\frac{\psi_3'}{3\psi_3}-\frac{\psi_2'}{2\psi_2}\right]-\frac{1}{3(2\psi_2)^{\frac{1}{2}}}\left[\alpha_1+\frac{\gamma_1}{\gamma_0}-\frac{3\psi_3}{2\psi_2}\right],\end{aligned}\right\} \tag{4.15}$$

where primes denote differentiation with respect to x. As indicated above, $A^{(0)}$ and $A^{(3)}$ are functions both of $\tau = t-t^*$ and t^* (or, equivalently, functions of t and t^*), since $x = x(t) = x(\tau+t^*)$, by (4.1). The factor J has been included here since Jq occurs in the integrand of Λ in (4.7) as a result of our 'normalization' of the Green's function.

Substituting then the expression (4.14) for Jq into that integrand we obtain, for the first correction term in (3.8),

$$\Theta_1 = \begin{cases}S_2^{(0)}(t,z;\,t^*)+S_2^{(3)}(t,z;\,t^*), & \text{for}\quad t > t^*,\\ 0, & \text{for}\quad t \leqslant t^*,\end{cases} \tag{4.16}$$

where

$$S_n^{(p)}(t,z;\,t^*) = -\frac{(n+1)^{n/(n+1)}}{\Gamma[1/(n+1)]}\int_{t'=t^*}^{t}\int_{z'=0}^{\infty}G_0(t,z;\,t',z')\,e^{-\zeta'^2}\times A^{(p)}(\tau',t^*)\frac{[(n+1)\,\zeta']^{[2p/(n+1)]}}{[(n+1)\,\tau']^{1/(n+1)}}dz'\,dt',$$

with

$$\zeta' = \zeta(z',\tau'),\quad \tau' = t'-t^*,$$

$\zeta(z,t)$ being defined in (4.9).

We have chosen to express the integrals arising here for $n = 2$ in terms of the more general integral $S_n^{(p)}$, since it has been found that integrals of this type arise frequently in the treatment of related expansion problems. Moreover, as shown in Appendix 1, the preceding double integral can be reduced to a single integral involving the confluent hypergeometric function, which for the case at hand, $n = 2$, becomes

$$S_2^{(p)} = -\left[\frac{3^{p/3}}{\Gamma(\frac{1}{3})}\right]^2 \Gamma\left(\frac{p+2}{3}\right) z \exp\{-\zeta^2\}$$
$$\times \int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)\left(\frac{p-2}{3}\right) A^{(p)}(\lambda\tau, t^*) \exp\{-(\lambda/1-\lambda)\zeta^2\}$$
$$\times {}_1F_1\left[\left(\frac{p+2}{3}\right); \frac{4}{3}; \left(\frac{\lambda\zeta^2}{1-\lambda}\right)\right] d\lambda. \tag{4.17}$$

Thus, equations (4.16) and (4.17) provide a closed-form expression for the first correction term to the asymptotic temperature profile for $Pe \to \infty$.

In practice, the integrals in (4.17) would have to be evaluated numerically due to the general complexity of the functions $A^{(p)}$ of (4.14). Usually, however, one is mostly interested in the resulting correction term to the heat-transfer rate, which is proportional to the surface-normal derivative of Θ_1. This derivative is given by (4.16) as

$$\lim_{z\to 0} \frac{\partial \Theta_1}{\partial z} = \begin{cases} \lim_{z\to 0}\left[\dfrac{\partial S_2^{(0)}}{\partial z} + \dfrac{\partial S_2^{(3)}}{\partial z}\right], & \text{for} \quad t > t^*, \\ 0, & \text{for} \quad t \leqslant t^*, \end{cases} \tag{4.18}$$

which can be further simplified because, as shown also in Appendix 1,

$$\lim_{z\to 0} \frac{\partial S_2^{(p)}}{\partial z} = -\left[\frac{3^{p/3}}{\Gamma(\frac{1}{3})}\right]^2 \Gamma\left(\frac{p+2}{3}\right) \int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{(p-2)/3} A^{(p)}(\lambda\tau, t^*)\, d\lambda. \tag{4.19}$$

Equations (4.18) and (4.19) embody the principal result of this paper, providing a closed-form expression for the first correction term to the asymptotic heat-transfer rate for $Pe \to \infty$. Again, it should be emphasized that this result is meaningful only if used in conjunction with a surface-temperature distribution $\Theta_s(x)$ that is continuous everywhere with, in general, $\Theta_s(0) = 0$. This is so because our regular perturbation analysis has excluded any treatment of equations such as (3.11*a*) which, as mentioned before, apply within a small region near the point of surface-temperature discontinuity. We turn now to a specific application of the above result.

5. Asymptotic expansion of the Nusselt number for $Pe \gg 1$, for an isothermal sphere in Stokes flow

The results derived above will now be applied to a problem of some historical interest (Acrivos & Taylor 1962) involving heat transfer from a solid sphere in steady Stokes flow. That is, we consider the case where

$$Re = U_\infty L/\nu \ll 1,$$

but with Pr sufficiently large that

$$Pe = U_\infty L/\kappa = Re\, Pr \gg 1,$$

where L denotes now the radius of the sphere and U_∞ the free-stream velocity of the fluid relative to the sphere.

The stream function for the sphere is given to a first approximation in Re by the well-known expression of Stokes (Lamb 1945)

$$\psi = \frac{1}{2}\left(1-\frac{3}{2r}+\frac{1}{2r^3}\right) r^2 \sin^2\phi + O(Re) \quad (0 \leqslant \phi \leqslant \pi), \tag{5.1}$$

for $Re \to 0$, where $LU_\infty \psi$ is the stream function, Lr is the radial distance from the centre of the sphere, and ϕ is the angle measured from the upstream stagnation streamline. The flow is symmetric about the axis of the sphere lying in the direction of the free-stream flow.

In this case, the co-ordinates r, ϕ are related to the boundary-layer co-ordinates x, y of § 2 by

$$\left.\begin{aligned} r &= 1+y, \\ \phi &= x \quad (0 \leqslant x \leqslant \pi), \\ \alpha &= r = 1+y; \quad \alpha_0 = \alpha_1 = 1, \\ \gamma &= r\sin\phi = (1+y)\sin x; \quad \gamma_0 = \gamma_1 = \sin x, \end{aligned}\right\} \tag{5.2}$$

so that, using (5.1), we have for the expansion (3.4) for ψ

$$\left.\begin{aligned} \psi &= \tfrac{1}{4}(3y^2 - y^3 + y^4 + \ldots)\sin^2 x, \\ \text{with} \qquad \psi_2(x) &= -3\psi_3(x) = \tfrac{3}{4}\sin^2 x. \end{aligned}\right\} \tag{5.3}$$

Consequently, the variables t and z of (4.1) become, for $n = 2$,

$$\left.\begin{aligned} z &= (2\psi_2)^{\frac{1}{2}} y_2 = (\sqrt{\tfrac{3}{2}}\sin x)\, y_2, \\ t &= \int_0^x (2\psi_2(s))^{\frac{1}{2}} \gamma_0(s)\, ds = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}(x - \tfrac{1}{2}\sin 2x), \end{aligned}\right\} \tag{5.4}$$

with $0 \leqslant t \leqslant t_m$ for $0 \leqslant x \leqslant \pi$, where $t_m = \frac{1}{2}\pi\sqrt{\frac{3}{2}}$. (It will be noted that the Jacobian of (4.3) is given by

$$J = \tfrac{2}{3}\sin^{-3} x,$$

indicating that the transformation is singular at $x = 0$ and at $x = \pi$.)

Now, for the case of uniform surface temperature, the boundary conditions on the (dimensionless) temperature field are

$$\left.\begin{aligned} \lim_{r\to 1} \Theta &= 1 \quad \text{for} \quad 0 \leqslant \phi \leqslant \pi, \\ \lim_{r\to\infty} \Theta &= 0 \quad \text{for} \quad 0 \leqslant \phi \leqslant \pi. \end{aligned}\right\} \tag{5.5}$$

Then, if the local Nusselt number (based on the radius of the sphere) is defined by

$$Nu(x) = -\left(\frac{\partial\Theta}{\partial r}\right)_{r=1} = -\left(\frac{\partial\Theta}{\partial y}\right)_{y=0}, \tag{5.6}$$

we obtain from equation (3.8) the asymptotic expansion

$$\begin{aligned} Nu(x) &= -(Pe)^{\frac{1}{3}}\left[\frac{\partial\Theta_0}{\partial y_2} + (Pe)^{-\frac{1}{3}}\frac{\partial\Theta_1}{\partial y_2} + O(Pe^{-\frac{2}{3}})\right]_{y_2=0} \\ &= -(Pr)^{\frac{1}{3}}\sqrt{\tfrac{3}{2}}\sin x\left[\frac{\partial\Theta_0}{\partial z} + (Pe)^{-\frac{1}{3}}\frac{\partial\Theta_1}{\partial z} + O(Pe^{-\frac{2}{3}})\right]_{z=0}, \end{aligned} \tag{5.7}$$

the last equality following from (5.4). We can now employ the general results derived above to give the first two terms in (5.7).

For a uniform surface temperature $x^* = t^* = 0$ in (4.11). Equation (4.12) gives then the asymptotic temperature profile as

$$\left.\begin{aligned} \Theta_0 &= \frac{\Gamma(\frac{1}{3},\zeta^2)}{\Gamma(\frac{1}{3})} = \frac{\Gamma(\frac{1}{3},z^3/9t)}{\Gamma(\frac{1}{3})}, \\ \text{so that} \qquad -\left(\frac{\partial\Theta_0}{\partial z}\right)_{z=0} &= \frac{1}{\Gamma(\frac{1}{3})}\left(\frac{3}{t}\right)^{\frac{1}{3}}, \end{aligned}\right\} \tag{5.8}$$

where $t(x)$ is to be obtained from (5.4). Equation (5.8) is identical with the well-known result for the asymptotic Nusselt number for $Pe \to \infty$ (cf. Acrivos & Taylor 1962).

In order to derive now the correction of $O(Pe^{-\frac{1}{3}})$ relative to the first term, we employ (4.18) and (4.19). For the problem at hand, the functions $A^{(0)}$, $A^{(3)}$ of (4.15) reduce, by (5.2) and (5.3), to

$$\begin{gathered} A^{(0)} = A^{(0)}(t) = 2/(2\psi_2)^{\frac{1}{2}}, \\ A^{(3)} = A^{(3)}(t) = \frac{1}{6}\left[\frac{\psi_2' t}{\gamma_0(2\psi_2)^2} - \frac{5}{(2\psi_2)^{\frac{1}{2}}}\right], \end{gathered} \tag{5.9}$$

where γ_0 and ψ_2 are given respectively by (5.2) and (5.3). As indicated, the functions in (5.9) depend now on t alone, since $t^* = 0$.

It is of some interest to note at this point that, by (5.2) to (5.4), one can readily show that t and z, as well as both the functions in (5.9) are all *odd* functions of x. Hence, it follows, by (4.17) and (5.8), that $\partial\Theta_i/\partial x = 0$ at $x = 0$ for $i = 0, 1$, which agrees of course with the exact 'initial' condition for the problem at hand. In turn this result establishes the validity in this case of the first two terms of (3.8) at $x = 0$, in spite of the singularity of the co-ordinate transformation.

By employing (4.18) and (4.19), we can derive immediately the desired correction term in (5.7), which becomes

$$\begin{aligned} -\left(\frac{\partial\Theta_1}{\partial z}\right)_{z=0} = \left[\frac{1}{\Gamma(\frac{1}{3})}\right]^2 \Gamma(\tfrac{2}{3}) \int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{\frac{2}{3}} A^{(0)}(\lambda t)\, d\lambda \\ + \left[\frac{3}{\Gamma(\frac{1}{3})}\right]^2 \Gamma(\tfrac{5}{3}) \int_0^1 \lambda^{\frac{4}{3}}(1-\lambda)^{\frac{1}{3}} A^{(3)}(\lambda t)\, d\lambda. \end{aligned} \tag{5.10}$$

Unfortunately it appears difficult to obtain an analytic expression for this correction term due to the complexity of the functions $A^{(0)}$ and $A^{(3)}$. However, one can deduce certain of its properties. In particular, it is shown in Appendix 2 that this expression has a logarithmic singularity at the rear stagnation point of the sphere $x = \pi\,(t = t_m)$. It follows then that the expansion of (5.7) cannot be valid in the neighbourhood of this point since the first term, which is $O(1)$ there, becomes much smaller than the 'correction' term as $t \to t_m$. This affords a good example of the non-uniform character of such expansions due, in this case, to the omission of conduction terms which govern the 'thickening' of the thermal boundary layer into a 'thermal wake'. Nevertheless, it is possible to obtain a simple result for the average Nusselt number for the entire surface of the sphere:

$$\overline{Nu} = \frac{\int_0^\pi Nu(x)\sin x\,dx}{\int_0^\pi \sin x\,dx} = \frac{1}{2}\int_0^\pi Nu(x)\sin x\,dx, \tag{5.11}$$

where $Nu(x)$ is the local Nusselt number of (5.6), since, as discussed in Appendix 2, the contribution from the singular region described above does not affect the correction term of $O(Pe^{-\frac{1}{3}})$ in the expansion of $\overline{Nu}/Pe^{\frac{1}{3}}$. The latter can be expressed as

$$\overline{Nu}/Pe^{\frac{1}{3}} = C_0 + C_1(Pe)^{-\frac{1}{3}} + o(Pe^{-\frac{1}{3}}), \tag{5.12}$$

where the C_k are numerical constants given by (5.7) and (5.11):

$$C_k = \frac{1}{2}\sqrt{\frac{3}{2}}\int_0^\pi \sin^2 x\left[\frac{\partial\Theta_k}{\partial z}\right]_{z=0} dx = \frac{1}{2}\int_0^{t_m}\left[\frac{\partial\Theta_k}{\partial z}\right]_{z=0} dt, \quad k = 0, 1,$$

t being the variable defined in (5.4).

By equation (5.8), the first constant in (5.12) is found to be (cf. Acrivos & Taylor 1962)

$$\left.\begin{aligned} C_0 = \frac{1}{2}\frac{1}{\Gamma(\frac{1}{3})}\int_0^{t_m}\left(\frac{3}{t}\right)^{\frac{1}{3}} dt &= \frac{3^{\frac{4}{3}}}{4\Gamma(\frac{1}{3})}[\sqrt{\tfrac{3}{2}}\tfrac{1}{2}\pi]^{\frac{2}{3}} \\ &= 0{\cdot}6245\ldots, \end{aligned}\right\} \tag{5.13}$$

whereas from equation (5.10)

$$C_1 = \tfrac{1}{2}[\Gamma(\tfrac{2}{3})/\{\Gamma(\tfrac{1}{3})\}^2]\,[B^{(0)} + B^{(3)}],$$

where

$$\left.\begin{aligned} B^{(0)} &= \int_{t=0}^{t_m}\int_{\lambda=0}^{1} \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}} A^{(0)}(\lambda t)\,d\lambda\,dt, \\ B^{(3)} &= 6\int_{t=0}^{t_m}\int_{\lambda=0}^{1} \lambda^{\frac{1}{3}}(1-\lambda)^{\frac{1}{3}} A^{(3)}(\lambda t)\,d\lambda\,dt, \end{aligned}\right\} \tag{5.14}$$

$A^{(0)}$ and $A^{(3)}$ being defined by (5.9).

Now, the first integral $B^{(0)}$ can be evaluated analytically as follows. Noting that

$$A^{(0)}(t) = 2/(2\psi_2)^{\frac{1}{2}} = 2\gamma_0(dx/dt) = 2(dF/dt)\ (\text{say}) \tag{5.15}$$

and changing the order of integration in $B^{(0)}$, we have

$$B^{(0)} = 2\int_0^1 \lambda^{-\frac{2}{3}}(1-\lambda)^{-\frac{2}{3}} F(\lambda t_m)\,d\lambda, \tag{5.16}$$

where

$$F(t) = \int_0^{x(t)} \gamma_0(x)\,dx = \int_0^x \sin x\,dx = 1 - \cos x.$$

Changing next the variable of integration in the integral of (5.16) to $1-\lambda$ gives

$$B^{(0)} = 2\int_0^1 \lambda^{-\frac{2}{3}}(1-\lambda)^{-\frac{2}{3}} F[(1-\lambda)\,t_m]\,d\lambda,$$

which, taken together with that integral, results in the symmetric integral

$$B^{(0)} = \int_0^1 (1-\lambda)^{-\frac{2}{3}} \lambda^{-\frac{2}{3}} [F\{\lambda t_m\} + F\{(1-\lambda)\, t_m\}]\, d\lambda.$$

In addition, from the definition of $F(t)$ in (5.15) and the definitions of t and t_m in (5.4), it follows that

$$F\{(1-\lambda)\, t_m\} + F\{\lambda t_m\} = 2 \quad \text{for} \quad 0 \leqslant \lambda \leqslant 1, \tag{5.17}$$

so that the preceding integral can be expressed simply as

$$B^{(0)} = 2\int_0^1 \lambda^{-\frac{2}{3}}(1-\lambda)^{-\frac{2}{3}}\, d\lambda = \frac{2[\Gamma(\frac{1}{3})]^2}{\Gamma(\frac{2}{3})} \tag{5.18}$$

on using the well-known expression for the beta function.

The second integral $B^{(3)}$ of (5.14) can also be simplified by a somewhat similar procedure. Noting that, by (5.15)

$$\frac{d^2F}{dt^2} = \frac{dx}{dt}\frac{d}{dx}(2\psi_2)^{-\frac{1}{2}} = -\frac{1}{\gamma_0}\frac{\psi_2'}{(2\psi_2)^2}$$

we have, because of the definition of $A^{(3)}$ in (5.9) and because of (5.15),

$$A^{(3)}(t) = -\frac{1}{6}\left[t\frac{d^2F}{dt^2} + 5\frac{dF}{dt}\right] = -\frac{1}{6}\frac{d}{dt}\left[t\frac{dF}{dt} + 4F\right],$$

which, with (5.15), gives readily

$$B^{(3)} = -\frac{10}{3}\int_0^1 \lambda^{-\frac{2}{3}}(1-\lambda)^{\frac{1}{3}} F(\lambda t_m)\, d\lambda - \frac{1}{3}\int_0^1 [\lambda^{-\frac{2}{3}}(1-\lambda)^{\frac{1}{3}} + \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}}]\, F(\lambda t_m)\, d\lambda.$$

Now, the second integral here can be written in the symmetric form

$$\frac{1}{2}\int_0^1 [\lambda^{-\frac{2}{3}}(1-\lambda)^{\frac{1}{3}} + \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}}]\, [F\{\lambda t_m\} + F\{(1-\lambda)\, t_m\}]\, d\lambda$$
$$= 2\int_0^1 \lambda^{-\frac{2}{3}}(1-\lambda)^{\frac{1}{3}}\, d\lambda = \frac{[\Gamma(\frac{1}{3})]^2}{\Gamma(\frac{2}{3})},$$

the equalities following, as with (4.18), from the application of (5.17). Thus, we obtain finally for the integral $B^{(3)}$ of (5.14)

$$B^{(3)} = -\frac{1}{3}\left[10\int_0^1 F(\lambda t_m)\, \lambda^{-\frac{2}{3}}(1-\lambda)^{\frac{1}{3}}\, d\lambda + \frac{[\Gamma(\frac{1}{3})]^2}{\Gamma(\frac{2}{3})}\right], \tag{5.19}$$

where the definite integral was found to equal $1{\cdot}185$ upon numerical integration.

Therefore, by (5.18) and (5.19),

$$C_1 = \tfrac{5}{6} - \tfrac{10}{6}(1{\cdot}185)\, \Gamma(\tfrac{2}{3})/[\Gamma(\tfrac{1}{3})]^2 \doteq 0{\cdot}461,$$

which, together with (5.12) and (5.13), gives the desired asymptotic relation for the average Nusselt number:

$$\overline{Nu} = (Pe)^{\frac{1}{3}}\, [0{\cdot}6245 + 0{\cdot}461 (Pe)^{-\frac{1}{3}} + O(Re) + o(Pe^{-\frac{1}{3}})] \tag{5.20}$$

for $Pe \to \infty$, $Re \to 0$. In closing, we recall that in this relation both $\overline{Nu}$ and Pe are based on the radius of the sphere.

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Appendix 1. Derivation of the integrals for the first correction term to heat transfer

By substituting the expression of (4.9) for the Green's function G_0 into the integrand of the integral $S_n^{(p)}$ of (4.16), and after changing the variables of integration, we obtain

$$S_n^{(p)} = -2\frac{(n+1)^{[(4p+1)/(n+1)]-1}}{\Gamma[1/(n+1)]}\int_0^t \frac{(\tau\tau')^{\frac{1}{2}}}{\tau-\tau'} A^{(p)}(\tau', t^*)\, e^{-a^2\zeta^2} F(\tau')\, d\tau',$$

where
$$F(\tau') = \int_0^\infty e^{-a^2 s}\, s^{[(2p+3)/(n+1)]-1} I_{1/(n+1)}(bs)\, ds, \tag{A 1.1}$$

$$a^2 = \tau/(\tau-\tau')$$

and
$$b = \frac{2(\tau\tau')^{\frac{1}{2}}}{(n+1)^{\frac{1}{2}}(\tau-\tau')}\zeta,$$

ζ and τ being the variables defined by (4.9) in terms of t, z, and t^*. To simplify the integral F which appears here, we make use of the relation (Erdelyi, Magnus, Oberhettinger & Tricomi 1954, vol. 2)

$$\int_0^\infty e^{-a^2 s^2} s^{\mu-1} I_\nu(bs)\, ds = \frac{\Gamma[\frac{1}{2}(\mu+\nu)]\,(\frac{1}{2}b/a)^\nu}{2a^\mu\,\Gamma(\nu+1)}\, {}_1F_1(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu;\, \nu+1;\, \tfrac{1}{4}b^2/a^2) \tag{A 1.2}$$

(valid when the real parts of a^2, $\nu+\mu$, and $\nu+1$ are positive) where ${}_1F_1$ denotes the confluenct hypergeometric function (Erdelyi *et al.* 1954, vol. 3). Furthermore, on account of the transformation rule

$${}_1F_1(\alpha;\beta;x) = e^x\, {}_1F_1(\beta-\alpha;\beta;-x) \tag{A 1.3}$$

equations (A 1.1)–(A 1.3) give for $S_n^{(p)}$

$$S_n^{(p)} = -\left[\frac{(n+1)^{p/(n+1)}}{\Gamma[1/(n+1)]}\right]^2 \Gamma\left(\frac{p+2}{n+1}\right) z\, e^{-\zeta^2}\int_0^1 \lambda^{1/(n+1)}(1-\lambda)^{(p-n)/(n+1)}$$
$$\times A^{(p)}(\lambda\tau, t^*)\, e^{-[\lambda/(1-\lambda)]\zeta^2}\, {}_1F_1\left(\frac{p+2}{n+1};\frac{n+2}{n+1};\frac{\lambda\zeta^2}{1-\lambda}\right) d\lambda, \tag{A 1.4}$$

after a change of the variable of integration from τ' to $\lambda = \tau'/\tau$.

Recalling by (4.9) that $\zeta^2 \propto z^{n+1}$, one arrives finally at

$$-\left[\frac{\partial S_n^{(p)}}{\partial z}\right]_{z=0} = \left[\frac{(n+1)^{p/(n+1)}}{\Gamma[1/(n+1)]}\right]^2 \Gamma\left(\frac{p+2}{n+1}\right)\int_0^1 \lambda^{1/(n+1)}(1-\lambda)^{(p-n)/(n+1)}$$
$$\times A^{(p)}(\lambda\tau, t^*)\, d\lambda \quad \text{for} \quad n > -1. \tag{A 1.5}$$

Although we shall not establish rigorously all the criteria for validity of this formal result, we note that two requirements for convergence of the integral in (A 1.5) are the integrability of $t^{1/(n+1)}A^{(p)}(t, t^*)$, regarded as a function of t over the interval $0 \leqslant t \leqslant t_m$, and the condition that $(p-n)/(n+1) > -1$ which reduces to $p > -1$ since we are considering here only the case $n > 0$.

Appendix 2. Singular behaviour of the asymptotic expansion for the sphere

We wish here to indicate briefly the singular nature of the expansion for the local rate of heat transfer, derived in the last section of the paper, for the case of the isothermal sphere. To begin with, one can see from equations (5.7) and (5.10) that the first correction term to the asymptotic value of the local Nusselt number is proportional to

$$-\left[\frac{\partial\Theta_1}{\partial z}\right]_{z=0} = \frac{\Gamma(\frac{2}{3})}{[\Gamma(\frac{1}{3})]^2}[F_1(t)+F_2(t)], \tag{A 2.1}$$

where

$$F_1(t) = \int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}}\,A^{(0)}(\lambda t)\,d\lambda, \tag{A 2.2}$$

$$F_2(t) = 6\int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{\frac{1}{3}}\,A^{(3)}(\lambda t)\,d\lambda,$$

with $A^{(0)}$ and $A^{(3)}$ given by (5.3) and (5.9) as

$$A^{(0)}(t) = 2\sqrt{\tfrac{2}{3}}/\sin x,$$

$$A^{(3)}(t) = \frac{1}{6}\left[\frac{2t}{3}\frac{\cos x}{\sin^4 x} - \frac{5\sqrt{\frac{2}{3}}}{\sin x}\right], \tag{A 2.3}$$

the functional relationship between x and t being that of (5.4).

It is immediately evident from the above expressions that F_1 and F_2 are both well behaved for all values of t with the exception of $t = 0$ and $t = t_m = \frac{1}{2}\sqrt{\frac{3}{2}}\pi$ which correspond, respectively, to the front and the rear stagnation points of the sphere. The singularity at $t = 0$ offers no real difficulties for, although it is true that both F_1 and F_2 become $O(t^{-\frac{1}{3}})$ at $t \to 0$, the same also holds for $(\partial\Theta_0/\partial z)_{z=0}$ on account of (5.8). Besides, as was remarked earlier following equation (5.9), our solution remains well behaved as t (or x) $\to 0$. In contrast, it will be shown presently that F_1 and F_2 both possess logarithmic singularities at the rear stagnation point of the sphere, thus indicating a breakdown, as $t \to t_m$, of the regular perturbation expansion since $(\partial\Theta_0/\partial z)_{z=0}$ remains $O(1)$.

Considering first the integral $F_1(t)$ of (A 2.2), one can see from (A 2.3) and (5.3) that the function $A^{(0)}$ in the integrand behaves like

$$\begin{aligned} A^{(0)}(t) &= [2\sqrt{\tfrac{2}{3}}/(\pi - x)]\,[1 + O\{(\pi-x)^2\}] \\ &= [2/3^{\frac{1}{3}}(t_m - t)^{\frac{1}{3}}]\,[1 + O\{(t_m-t)^{\frac{2}{3}}\}] \quad \text{for} \quad x \to \pi,\, t \to t_m. \end{aligned}$$

Now, the first integral of (A 2.2) can be expressed as

$$F_1(t) = 2/3^{\frac{1}{3}}\int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}}\,(t_m - \lambda t)^{-\frac{1}{3}}\,d\lambda$$
$$+\int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}}\,[A^{(0)}(\lambda t) - (2/3^{\frac{1}{3}})\,(t_m - \lambda t)^{-\frac{1}{3}}]\,d\lambda\ldots, \tag{A 2.4}$$

from which it follows that, as $t \to t_m$, the second integral here remains bounded, whereas the first integral has a singularity at the upper limit, $\lambda = 1$. In fact, one can show readily that

$$\int_0^1 \lambda^{\frac{1}{3}}(1-\lambda)^{-\frac{2}{3}}(t_m - t\lambda)^{-\frac{1}{3}}\,d\lambda = t_m^{-\frac{1}{3}}\log\left[1/(t_m - t)\right] + O(1)$$
$$= t_m^{-\frac{1}{3}}\log\left[1/(\pi - x)\right] + O(1), \quad \text{for} \quad x \to \pi, t \to t_m,$$

so that the behaviour near $x = \pi$ of the function $F_1(t)$ of (A 2.1) is given by

$$F_1(t) = (2t_m^{-\frac{1}{3}}/3^{\frac{1}{6}})\log\left[1/(t_m - t)\right] + O(1)$$
$$= (2^{\frac{3}{2}}/(3\pi)^{\frac{1}{3}})\log\left[1/(\pi - x)\right] + O(1) \quad \text{for} \quad x \to \pi, t \to t_m. \qquad (\text{A } 2.5)$$

In a similar manner one can demonstrate that the function $F_2(t)$ of (A 2.1) has the limiting form

$$F_2(t) = -(t_m^{-\frac{1}{3}}/2\cdot 3^{\frac{1}{3}})\log\left[1/(t_m - t)\right] + O(1)$$
$$= -(1/6\pi^{\frac{1}{3}})\log\left[1/(\pi - x)\right] + O(1) \quad \text{for} \quad x \to \pi, t \to t_m. \qquad (\text{A } 2.6)$$

Thus, because of (A 2.1), (A 2.5) and (A 2.6),

$$-\left[\frac{\partial\Theta_1}{\partial z}\right]_{z=0} = \frac{\Gamma(\frac{2}{3})}{[\Gamma(\frac{1}{3})]^2}\left(\frac{2}{3}\right)^{\frac{7}{6}}\left(\frac{2}{\pi}\right)^{\frac{1}{3}}(6^{\frac{1}{3}} - 1)\log\left[1/(\pi - x)\right] + O(1)\ldots \qquad (\text{A } 2.7)$$

for $x \to \pi$.

We can see then that, since $(\partial\Theta_0/\partial z)_{z=0}$ is $O(1)$ as $x \to \pi$, the asymptotic expansion (5.7) is not uniformly valid for $0 \leqslant x \leqslant \pi$, due to the singularity at $x = \pi$. This singularity is clearly attributable to the omission, in going from (3.2) to (3.7), of the term $\partial^2/\partial x^2$ from the differential operator P which results in the boundary-layer solution not being valid near $x = \pi$ where this derivative becomes important in describing the coalescence of the thermal boundary layers into a thermal wake behind the sphere.

Now, by investigating the limiting form of the differential operator of (3.2) near $x = \pi$, one can show that in order to retain the x-derivative in question one should employ as appropriate stretched co-ordinates for the boundary-layer analysis $(Pe)^{\frac{1}{3}}y$, as before, and $(Pe)^{\frac{1}{3}}(\pi - x)$, in which case the 'singular' region at the rear stagnation point will extend over a neighbourhood, $(\pi - x) = O(Pe^{-\frac{1}{3}})$, of $x = \pi$. Therefore, by inspection of the integral (5.11) for the average Nusselt number, one can conclude that the contribution of the singular region to the total rate of heat transfer should be of an order smaller than $O(Pe^{-\frac{1}{3}})$ relative to the asymptotic value for $Pe \to \infty$, and that the expansion of (5.12) should be correct to terms $O(Pe^{-\frac{1}{3}})$, which is as indicated.

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